On Magnetic solution to 2+1 Einstein–Maxwell gravity

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The three–dimensional magnetic solution to the Einstein–Maxwell field equations have been considered by some authors. Several interpretations have been formulated for this magnetic spacetime. Up to now this solution has been considered as a two–parameter self–consistent field. We point out that the parameter related to the mass of this solution is just a pure gauge and can be rescaled to minus one. This implies that the magnetic metric has really a simple form and it is effectively one-parameter solution, which describes a distribution of a radial magnetic field in a 2+1 anti–de Sitter background space–time. We consider an alternative interpretation to the Dias–Lemos one for the magnetic field source.

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The 2+1 magnetic solution to the Einstein–Maxwell field equations has been studied by some authors. The static solution has been found by Clement [1], Peldán [2], Hirschmann and Welch [3] and Cataldo and Salgado [4], using different procedures. The generalization to the rotating case was done by Dias and Lemos [5]. This solution may be written in the form

$$ds^2 = -\left(\frac{r^2}{l^2} - M\right) dt^2 + \frac{r^2 dr^2}{\left(\frac{r^2}{l^2} - M\right)\left(r^2 + Q_m^2 \ln\left|\frac{r^2}{l^2} - M\right|\right)} + \left(r^2 + Q_m^2 \ln\left|\frac{r^2}{l^2} - M\right|\right) d\phi^2, \qquad (1)$$

where l is the radius of a pseudo–sphere related to the cosmological constant via $l = -1/\sqrt{\Lambda}$, Q_m and M are self–consistent integration constants of the Einstein–Maxwell field equations. The vector potential 1–form of this gravitational field is given by

$$A = \frac{Q_m}{2} \ln\left|\frac{r^2}{l^2} - M\right| d\phi.$$

When $Q_m = 0$ the metric (1) reduces to the nonrotating three–dimensional Bañados–Teitelboim–Zanelli black hole [6], where M is the mass of this uncharged metric, which has an event horizon at $r = \sqrt{M} l$. Let us study the behavior of this Einstein–Maxwell field. We shall consider the values of the r–coordinate for which the component $g_{\phi\phi}$ becomes zero. This occurs to be for some value of $r = \overline{r}$, which satisfies the constraint

$$\overline{r}^{\,2} + Q_m^2 \ln\left|\frac{\overline{r}^{\,2}}{l^2} - M\right| = 0. \qquad (2)$$

This equation implies that $\overline{r}$ is constrained to be between

$$l\sqrt{M} < \overline{r} \leq l\sqrt{M+1}. \qquad (3)$$

The metric (1) appears to change the signature at $r = \overline{r}$. This indicates us that we are using an incorrect extension. The correct one can be found setting

$$x^2 = r^2 - \overline{r}^{\,2}, \qquad (4)$$

since the physical space–time has sense only for $r \geq \overline{r}$ and we have $0 \leq x < \infty$. Taking into account the constraint (2), the metric (1) becomes

$$ds^2 = -\left(\frac{x^2}{l^2} + \frac{\alpha^2}{l^2}\right) dt^2 + \frac{l^2 x^2 dx^2}{(x^2+\alpha^2)F^2(x)} + F^2(x) d\phi^2, \qquad (5)$$

where $\alpha^2 = \overline{r}^2 - l^2 M$ and the function $F^2(x)$ is defined as

$$F^2(x) = x^2 + Q_m^2 \ln\left(1 + \frac{x^2}{\alpha^2}\right) \qquad (6)$$

This metric is horizonless, without curvature singularities and in particular, there is no a magnetically charged three–dimensionally black hole [3]. The presented magnetic solution shows a strange behavior. As the parameter Q_m, related to the strength of the magnetic field, goes to zero we should recover the Bañados–Teitelboim–Zanelli black hole, but it does not occur. Since, in this case “the limit of a theory is not the theory of the limit”. Surprisingly, this strange behavior can be eliminated by

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introducing a new set of coordinates. Effectively, making the following rescaling transformations

$$t'^2 = \frac{\overline{r}^{\,2} - Ml^2}{l^2}\, t^2, \quad r'^2 = \frac{l^2}{\overline{r}^{\,2} - Ml^2}\, x^2,$$
$$\phi'^2 = \frac{\overline{r}^{\,2} - Ml^2}{l^2}\, \phi^2, \tag{7}$$

and introducing them into Eq. (5), we obtain the following metric

$$ds^2 = -\left(\frac{r'^{\,2}}{l^2} + 1\right) dt'^{\,2} + \frac{r'^{\,2} dr'^{\,2}}{(\frac{r'^{\,2}}{l^2} + 1) F'^{\,2}(r')} + F'^{\,2}(r') d\phi'^{\,2}, \tag{8}$$

where

$$F'^{\,2}(r') = r'^{\,2} + \tilde{q}_m^2 \ln\left(\frac{r'^{\,2}}{l^2} + 1\right), \tag{9}$$

and $\tilde{q}_m^2 = Q_m^2 e^{\overline{r}^{\,2}/Q_m^2}$. In the present form, the constant M has been eliminated and the metric (8) has one integration constant q_m. This parameter is well behaved for $\overline{r} = 0$ and the magnetic field can be switched off without any problem. When $\tilde{q}_m = 0$, the anti–de Sitter space is obtained. Clearly the metric (8) is a particular solution of the line element (1), where we need to put $M = -1$. This implies that the parameter related to the mass of this solution is just a pure gauge and it can be rescaled to the value -1. This agrees with the Dias–Lemos result [5], who have shown that the mass of the magnetic solution (5) is negative. However, the examined by authors three–dimensional static magnetic field is still a two–parameter solution, since the mass is considered a free parameter (see their Eq. (3.24) with $\Omega = J = Q_e = 0$). Thus, the metric (5) is really a one–parameter solution with a distribution of a radial magnetic field in a 2+1 anti–de Sitter background, which takes the form of Eq. (8). This metric can be considered as the general "physical solution" to the self–consistent problem for a superposition of a radial magnetic field and a 2+1 Einstein static gravitational field. Clearly the metric (8) is not a magnetically charged three–dimensionally black hole. This metric is horizonless (in this sense this is a particle–like solution), without curvature singularities and it has no signature change. The solution (8) does have a conical singularity at $r' = 0$ which can be removed by identifying the ϕ'–coordinate with the period $T_{\phi'} = 2\pi/(1 + \tilde{q}_m^2/l^2)$ [3]. It is well behaved, since if $\tilde{q}_m$ approaches infinity, this period becomes zero, while if $\tilde{q}_m$ approaches zero, this period goes to 2π, since the anti–de Sitter space has no angle deficit. Finally, let us consider an alternative interpretation of this magnetic solution. In the reference quoted above [5], the authors have shown that the magnetic field source can be neither a Nielson–Oleson vortex solution nor a Dirac monopole. Thus they attempted to provide an interpretation of this magnetic solution. Dias and Lemos interpreted the static magnetic field source as being composed by a system of two symmetric and superposed electric charges. One of the electric charges is at rest and the other is spinning around it. In view of the symmetry of the space–time this configuration is located at the origin of the coordinate system.

We propose here another interpretation based on the similarities of static Einstein–Maxwell theory for 2+1 dimensional rotationally symmetric spacetimes and 3+1 dimensional axially symmetric spacetimes [7]. Let us refer to the static magnetic fields in four dimensional general relativity. Bonnor [8], studying this topic, have considered axially symmetric magnetostatic gravitational fields in empty spaces generated from known electrostatic solutions. Bonnor noted here that when we generate magnetostatic solutions from electrostatic ones "there is not an equivalence between sources of the static electric and magnetic fields; by this is meant that whereas the electrostatic field in empty space may be considered to arise from point–charges, the magnetostatic field must arise from dipoles, or from stationary electric currents" [8]. The above remark may have profound implications for the nature of the studied 2+1 dimensional magnetic spacetime. Effectively, Bonnor generates a magnetostatic solution from a set of electrostatic ones, for electric fields containing no matter or charge except at singularities (see Eqs. (3.4) and (3.5) of the Ref. [8]). This solution has two constants of integration, representing the mass and the electric field strength. The generated 3+1 dimensional magnetostatic solution has physical sense only if we take zero the parameter representing the mass; then the solution is regarded as referring simply to a uniform magnetic field produced by a solenoid without mass [8]. The similarity that happen between 2+1 and 3+1 dimensions is clear: the three–dimensional magnetic solution may be generated from the electrostatic Bañados–Teitelboim–Zanelli black hole with the help of a duality mapping [4, 7]. In this case the electric field arises from a charged point mass (excluding interior solutions from consideration). As we have shown the three–dimensional magnetostatic gravitational field is really a one–parameter solution, where the free parameter is only the integration constant related to the magnetic field strength. Thus, the source of the magnetic field may be considered a two dimensional solenoid, i.e. a circular current. We prefer to locate this current at spatial infinity, since the curvature is regular everywhere.

We should note that the Bonnor solution with an uniform magnetic field is valid for the case in which the cosmological constant is vanished [8]. In our case the magnetic field is given by

$$B(r) \sim \frac{1}{\sqrt{\frac{r^2}{l^2} + 1}}, \tag{10}$$

and is regular everywhere. From this expression we see that the magnetic field at the origin has a maximum value, and at infinity approaches to zero. This magnetic field is not a constant since the cosmological constant is negative and then it acts as an attractive gravitational

force. This implies that the magnetic lines held together near the origin.

Note added. In a recently appeared work the thin shell collapse, leading to the formation of charged rotating black holes in 2+1 dimensions, is considered [9]. In this context, from physical considerations, the author singles out from the solution (1) the case $M = -1$, since for this choice of the parameter M the magnetic solution does not exhibit a pathological behavior. In this case a charged rotating thin shell is interpreted as the analog to a solenoid carrying a steady current, and then inside the thin shell the three dimensional $M = -1$ magnetic static solution is valid, and the magnetic field just vanishes outside the rotating thin shell.

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[1] G. Clement, Class. Quant. Grav. **10** (1993) L49.
[2] P. Peldán, Nucl. Phys. B **395** (1993) 239.
[3] E. Hirschmann and D. Welch, Phys. Rev. D **53** (1996) 5579.
[4] M. Cataldo and P. Salgado, Phys. Rev. D **54** (1996) 2971.
[5] O.J.C. Dias and J.P.S. Lemos, JHEP **01** (2002) 006.
[6] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. **69** (1992) 1849.
[7] M. Cataldo, Phys. Lett. B **529** (2002) 143.
[8] W.B. Bonnor, Proc. Phys. Soc. Lond. A **67** (1954) 225.
[9] R. Olea, Charged Rotating Black Hole Formation from Thin Shell Collapse in Three Dimensions, hep–th/0401109.